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LETTER TO THE EDITOR

New exact exponents for two-dimensional self-avoiding walks

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Abstract. We establish, using standard Coulomb gas methods, the values of a new set of geometrical exponents that we have previously conjectured. These exponents give, for any p, the number of configurations of p two-dimensional self-avoiding walks of the same length l which are attached by their ends:

$$\omega_{pl} \sim \mu^{pl} l^{(20-9p^2)/32-(p-1)}.$$

In an earlier paper (Saleur 1986, hereafter referred to as I) we have introduced new exponents γ_p for the two-dimensional self-avoiding walk (SAW) problem. If ω_{pl} is the number of configurations per lattice site for p non-intersecting SAW *each* of length l which are attached by their extremities (figure 1) we have defined γ_p by

$$\omega_{pl} \sim \mu^{pl} l^{\gamma_p - 1} \qquad l \to \infty \tag{1}$$

where μ is the connectivity constant of the lattice. (For p large, the walks cannot start at the same point but their extremities are tied together in a fixed neighbourhood.) The values of γ_1 and γ_2 are already known: $\gamma_1 = \gamma = \frac{43}{32}$ and $\gamma_2 = 1 - 2\nu = -\frac{1}{2}$. In order to determine γ_p for all p, we have considered operators ϕ_p which appear as composite operators in the $n = 0$ vector model (de Gennes 1979) and whose correlation function is

$$\langle \phi_p(R) \phi_p(R') \rangle = \sum_{\mathcal{G}_p} \beta^{l(\mathcal{G}_p)} \tag{2}$$

where the $\mathcal{G}_p$ are graphs formed by p SAW of *total* length l connecting points R and R'. The critical point of the $n = 0$ vector model is $\beta_c = \mu^{-1}$. At this value, (2) decays algebraically:

$$\langle \phi_p(R) \phi_p(R') \rangle \sim |R - R'|^{-2x_p}. \tag{3}$$

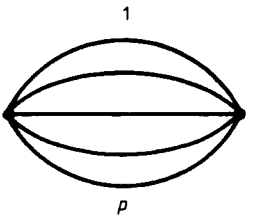

Figure 1. The 'watermelon' graph formed by p polymers attached by their ends. On a lattice, the polymers cannot start at the same point with no other intersection if p is large enough. In this case their extremities must be tied together in a fixed neighbourhood. This does not change the γ_p.

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Using several arguments of conformal invariance (see Belavin *et al* (1984) and references therein) and transfer matrix calculations (Cardy 1984) we have been led in I to conjecture the expression of x_p in 2D:

$$x_p = \frac{9p^2-4}{48} \qquad \forall p \geq 1 \tag{4}$$

generalising the known values $x_1 = \mathscr{I}_H = \frac{5}{48}$ and $x_2 = \mathscr{I}_T = \frac{2}{3}$. From (4) we have then deduced, by inversing the Laplace transform in (2),

$$\gamma_p - 1 = \frac{20-9p^2}{32} - (p-1). \tag{5}$$

In a series of recent papers, Duplantier (1986) has used these γ_p to predict several other results, like, for instance, the exponents for star polymers (Lipson *et al* 1985) or the contact exponents (Redner 1980), and found them to be in good agreement with numerical calculations. He has also calculated γ_p in the $\varepsilon = 4-d$ expansion, obtaining for $\varepsilon = 2$ a formula close to (5).

In this letter we would like to point out that the x_p can also be obtained in a more direct way using Coulomb gas arguments (José *et al* 1977) similar to those developed by Nienhuis (1982, 1984). For clarity we shall first recall briefly some important points of these arguments concerning the O(n) model (see Nienhuis (1982, 1984) for details).

The starting point is a generalised O(n) model on the honeycomb lattice with the partition function

$$Z_{O(n)} = \int \prod_R d\Omega_R(n) \prod_{(R'R'')} (1 + \beta s_{R'} \cdot s_{R''}) \tag{6}$$

where s is a n-component classical spin with $|s|^2 = n$ and $d\Omega(n)$ is the normalised n-dimensional angular measure. Using a high temperature expansion, (6) can be rewritten as

$$Z_{O(n)} = \sum_{\mathscr{G}_r} n^r \beta^{l(\mathscr{G}_r)} \tag{7}$$

where the $\mathscr{G}_r$ are graphs formed by r non-intersecting self-avoiding rings of total length l (if $n = 0$ these rings are forbidden and one recovers the SAW problem). One can show that $Z_{O(n)}$ is also the partition function of a solid-on-solid model on the dual triangular lattice (TSOS model). The heights are variable θ multiples of π. Two adjacent θ can differ by $0, \pm\pi$ and the weights are defined by considering triangles of three adjacent θ. If these are equal the weight of the triangle is 1; if one is larger than the two others the weight is $\beta \exp(iu)$ and if one is smaller than the two others the weight is $\beta \exp(-iu)$. By considering the polygons in (7) as walls between regions of equal θ one gets $Z_{O(n)} = Z_{TSOS}$ provided $n = 2\cos 6u$. Finally the TSOS model can be transformed into a six-vertex model on the covering Kagomé lattice and mapped onto a Coulomb gas. Nienhuis has deduced from this mapping the critical point $\beta_c(n)$ of (7) and the exponents $\mathscr{I}_H(n)$, $\mathscr{I}_T(n)$. These can be found in the original paper and we simply recall here the value of the renormalised coupling constant g at criticality:

$$n = -2\cos \pi g \qquad g \in [1, 2]. \tag{8}$$

We now introduce a generalisation of (2):

$$\langle \phi_p(R) \phi_p(R') \rangle = \frac{1}{Z_{O(n)}} \sum_{\mathscr{G}_{p,r}} n^r \beta^{l(\mathscr{G}_{p,r})} \tag{9}$$

where the $\mathcal{G}_{p,r}$ are graphs formed by p SAW connecting R to R' plus r self-avoiding rings (figure 2). At the critical temperature $\beta_c(n)$

$$\langle \phi_p(R)\phi_p(R')\rangle \sim |R-R'|^{-2x_p(n)} \tag{10}$$

and the x_p in (4) correspond to the case $n=0$. For calculating $x_p(n)$ we generalise the original determination by Nienhuis (1982) of $\mathcal{S}_H(n)=x_1(n)$. The basic idea is to consider the graphs $\mathcal{G}_{p,r}$ as representing dislocations in the TSOS model with a vortex at one extremity and an antivortex at the other extremity. Since a walk represents a wall of step π, the magnetic charge of these vertices is $m=\pm\frac{1}{2}p$. However the weights in the TSOS model associate a certain energy to the curvature of the walks so (9) is not exactly a vortex-antivortex correlation function. The main difference is that two walks of the same length have weights which differ by $\exp(\pm 6iu)$ for each turn around one of the endpoints. This can be compensated by adding to each vortex a spin wave $\exp[-(6iu/\pi)\theta]$ corresponding to an electric charge $e=-6u/\pi=1-g$. Suppose now there is a ring surrounding both endpoints. If its interior is raised (lowered) by π, it has a factor $\exp(6iu)$ ($\exp(-6iu)$). The two spin waves change those factors in $\exp(-6iu)$ ($\exp(6iu)$) so the total weight is left invariant. Except for some global phase factors, (9) is thus the correlation function of a combination of a vortex *and* a spin wave operator. By using the known renormalisation equations for fugacities of electric and magnetic charges (José *et al* 1977, Nienhuis 1982, 1984) one gets the dimension

$$x_p(n)=\frac{g}{2}m^2-\frac{1}{2g}e^2 \tag{11a}$$

or

$$x_p(n)=\frac{g}{2}\left(\frac{p}{2}\right)^2-\frac{1}{2g}(g-1)^2. \tag{11b}$$

If $n=0$, $g=\frac{3}{2}$ (8) and thus

$$x_p(0)=x_p=\frac{9p^2-4}{48} \tag{12}$$

in agreement with (4). Note that $x_2(n)=\mathcal{S}_T(n)$ only for $n=0$. Using the same method as in I we have checked (11) for general values of n using transfer matrix calculations. Our study thus confirms the conjecture (4) and gives a simple generalisation.

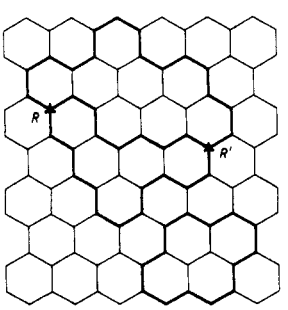

Figure 2. A typical graph of formula (9) with $r=1$, $p=3$ (bold lines are polymer links). In the TSOS language, it corresponds to a dislocation with a vortex at one extremity (say R) and an antivortex at the other one (R'). The magnetic charge of these vertices is $m=\pm\frac{3}{2}$. Thus, by describing a closed path around one extremity, the height varies by $\pm 2m\pi$.

In conclusion we should remark that (11) is a Kac (1979) formula in the conformal invariant theory of the O(n) model (Dotsenko and Fateev 1984). Let us consider, for instance, the case $n=1$ which has a central charge $C=\frac{1}{2}$ (Belavin *et al* 1984). The conformal dimensions of the primary operators in the degenerate representations are

$$h_{s,t}=\frac{(3s-4t)^2-1}{48} \tag{13}$$

with s, t integers $\geqslant 1$. It is then easy to identify the $x_p(1)$ as two series (13): $x_{2p'-1}(1)=2h_{2,p'+1}$ and $x_{2p'}(1)=2h_{4,p'+3}$ ($p'\geqslant 1$). Except for $x_1(1)=2h_{2,2}=\mathscr{D}_H(1)$, these dimensions are *outside* the minimal block $s\leqslant 3$, $t\leqslant 2$ (Belavin *et al* 1984). One can make a similar identification for other values of n. This thus provides a physical interpretation for some lines of the conformal grid in the O(n) models.

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